MAGNETIC FIELD

Lecture :Vector Potential

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Objectives

In this lecture you will learn the following

- Define vector potential for a magnetic field.
- Understand why vector potential is defined in a gauge.
- Calculate vector potential for simple geometries.
- Define electromotive force and state Faraday's law of induction

Vector Potential

For the electric field case, we had seen that it is possible to define a scalar function ϕ called the ``potential" whose negative gradient is equal to the electric field : $\nabla\phi = \vec{E}$. The existence of such a scalar function is a consequence of the conservative nature of the electric force. It also followed that the electric field is irrotational, i.e. $\text{curl } \vec{E} = \mathbf{0}$.

For the magnetic field, Ampere's law gives a non-zero curl

$$\text{curl } \vec{B} = \mu_0 \vec{J}$$

Since the curl of a gradient is always zero, we cannot express $\vec{B}$ as a gradient of a scalar function as it would then violate Ampere's law.

However, we may introduce a vector function $\vec{A}(\vec{r})$ such that

$$\vec{B} = \nabla \times \vec{A}$$

This would automatically satisfy $\nabla \cdot \vec{B} = 0$ since divergence of a curl is zero. $vecA$ is known as vector potential . Recall that a vector field is uniquely determined by specifying its divergence and curl. As $\vec{B}$ is a physical quantity, curl of $vecA$ is also so. However, the divergence of the vector potential has no physical meaning and consequently we are at liberty to specify its divergence as per our wish. This freedom to choose a vector potential whose curl is $\vec{B}$ and whose divergence can be conveniently chosen is called by mathematicians as a choice of a *gauge* . If ψ is a scalar function any transformation of the type

$$\vec{A} \longrightarrow \vec{A} + \nabla\psi$$

gives the same magnetic field as curl of a gradient is identically zero. The transformation above is known as *gauge invariance* . (we have a similar freedom for the scalar potential ϕ of the electric field in the sense that

it is determined up to an additive constant. Our most common choice of $\phi$is one for which $\phi \rightarrow 0$at infinite distances.)
A popular gauge choice for $vecA$is one in which

$$\nabla \cdot \vec{A} = 0$$

which is known as the ``Coulomb gauge". It can be shown that such a choice can always be made.

Exercise 1

Show that a possible choice of the vector potential for a constant magnetic field $\vec{B}$is given by $\vec{A} = (1/2)\vec{B} \times \vec{r}$. Can you construct any other $vecA$?

(Hint : Take $\vec{B}$in z-direction, express $vecA$in component form and take its curl.)

Biot-Savart's Law for Vector Potential

Biot-Savart's law for magnetic field due to a current element $\vec{dl}$

$$d\vec{B} = \frac{\mu_0 I}{4\pi}\frac{\vec{dl} \times \hat{r}}{r^2} = -\frac{\mu_0 I}{4\pi}\vec{dl} \times \nabla(\frac{1}{r})$$

may be used to obtain an expression for the vector potential. Since the element $\vec{dl}$does not depend on the position vector of the point at which the magnetic field is calculated, we can write $d\vec{B} = \frac{\mu_0 I}{4\pi}\nabla \times (\frac{\vec{dl}}{r})$

the change in sign is because $\nabla(\frac{\vec{dl}}{r}) = \nabla(1/r) \times \vec{dl}$.

Thus the contribution to the vector potential from the element $\vec{dl}$is $d\vec{A} = \frac{\mu_0 I}{4\pi r}\vec{dl}$

The expression is to be integrated over the path of the current to get the vector potential for the system

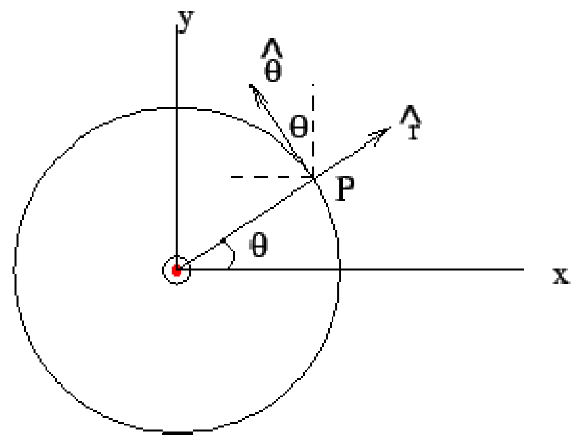
$$\vec{A} = \frac{\mu_0 I}{4\pi}\int\frac{\vec{dl}}{r}$$

Example 16

Obtain an expression for the vector potential at a point due to a long current carrying wire.
Solution :
Take the wire to be along the z-direction, perpendicular to the plane of the page with current flowing in a direction out of the page. The magnitude of the field at a point P is $\mu_0 I/2\pi r$with its direction being along the tangential unit vector $\hat{\theta}$at P, $\vec{B} = \frac{\mu_0 I}{2\pi r}\hat{\theta}$

The direction of $\vec{B}$makes an angle $(\pi/2) + \theta$with the x direction, where $\tan\theta = y/x$. Thus

$$\begin{aligned}\hat{\theta} &= \hat{\imath}\cos\left(\frac{\pi}{2}+\theta\right) + \hat{\jmath}\sin\left(\frac{\pi}{2}+\theta\right)\\ &= -\hat{\imath}\sin\theta + \hat{\jmath}\cos\theta\\ &= -\hat{\imath}\frac{y}{r} + \hat{\jmath}\frac{x}{r}\end{aligned}$$



Hence we have $\vec{B} = \frac{\mu_0 I}{2\pi}\frac{(-y\hat{\imath}+x\hat{\jmath})}{x^2+y^2}$

We wish to find a vector function $vecA$whose curl is given by the above. One can see that the following function fits the requirement

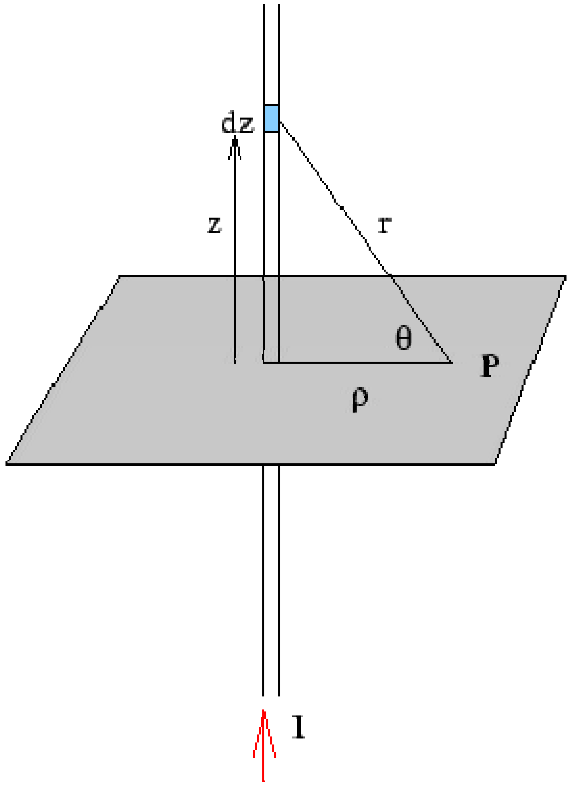
$$\vec{A} = -\frac{\mu_0 I}{4\pi}\ln(x^2+y^2)\hat{k} \qquad (1)$$

In the following, we will derive this directly from the expression for Biot-Savart's law. If $\rho$is the distance of P from an element of length dzat $z$of the wire, we have, $r^2 = x^2+y^2+z^2 = \rho^2+z^2$

Thus $\vec{A} = \frac{\mu_0 I}{4\pi}\int \frac{dz}{(\rho^2+z^2)^{1/2}}$

If the above integral is evaluated from $z=-\infty$to $z=+\infty$, it diverges. However, we can eliminate the infinity in the following manner. Let us take the wire to be of length $2L$so that

$$\vec{A} = \frac{\mu_0 I}{4\pi}\hat{k}\int_{-L}^{L}\frac{dz}{(\rho^2+z^2)^{1/2}}$$



The integral is evaluated by substituting $z = \rho\tan\theta$, so that $dz = \rho\sec^2\theta d\theta$. We get

$$\vec{A} = \frac{\mu_0 I}{4\pi}\hat{k}\int_{-\alpha}^{\alpha}\sec\theta d\theta$$
$$= \frac{\mu_0 I}{2\pi}\hat{k}\ln(\sec\alpha + \tan\alpha)$$

where $\tan\alpha = L/\rho$.

In terms of L and ρ, we have $\sec\alpha = \frac{(L^2+\rho^2)^{1/2}}{\rho} = \frac{L}{\rho}\left(1+\frac{\rho^2}{L^2}\right)^{1/2} \approx \frac{L}{\rho}\left(1-\frac{\rho^2}{2L^2}\right)$

Thus to leading order in L, $A = \frac{\mu_0 I}{2\pi}\hat{k}\ln(2L/\rho) = \frac{\mu_0 I}{2\pi}(\ln 2L - \ln\rho)\hat{k}$

As expected, for $L \longrightarrow \infty$, the expression diverges. However, since $vecA$ itself is not physical while curl of $vecA$ is, the constant term (which diverges in the limit of $L \longrightarrow \infty$) is of no consequence and $vecA$ is given by

$$A = -\frac{\mu_0 I}{2\pi}\hat{k}\ln\rho = -\frac{\mu_0 I}{4\pi}\ln(x^2+y^2)$$

which is the same as Eqn. (1)

Example 17

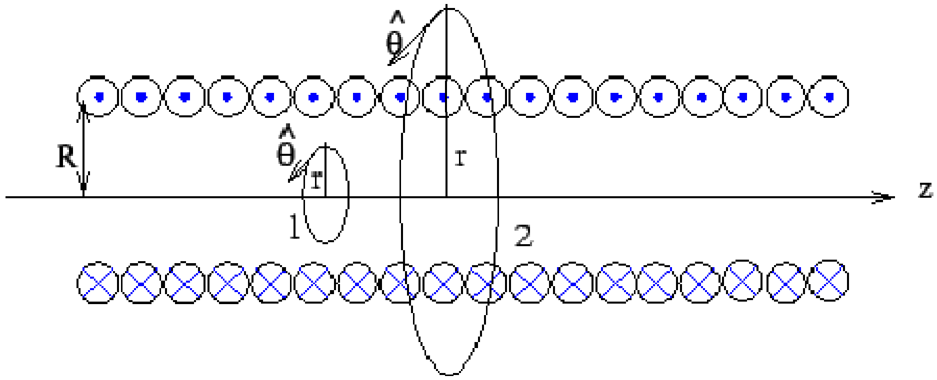
Obtain an expression for the vector potential of a solenoid.

Solution :

We had seen that for a solenoid, the field is parallel to the axis for points inside the solenoid and is zero outside.

$$\begin{aligned} B &= \mu_0 n I \hat{k} \text{ insidesolenoid} \\ &= 0 \text{outside} \end{aligned}$$

Take a circle of radius r perpendicular to the axis of the solenoid. The flux of the magnetic field is



$$\begin{aligned} \int \vec{B} \cdot d\vec{S} &= \mu_0 n I . \pi r^2 \quad \text{for} r \le R \\ &= \mu_0 n I . \pi R^2 \quad \text{for} r \ge R \end{aligned}$$

Since $\vec{B}$is axial, $vec A$is directed tangentially to the circle. Further, from symmetry, the magnitude of $vec A$is constant on the circumference of the circle.

Use of Stoke's theorem gives

$$\begin{aligned} \int \vec{B} \cdot d\vec{S} &= \int \text{curl } \vec{A} \cdot d\vec{S} \\ &= \oint \vec{A} \cdot d\vec{l} \\ &= \mid A \mid 2\pi r \end{aligned}$$

Thus

$$\begin{aligned} \vec{A} &= \frac{\mu_0 n I \pi r^2}{2\pi r} \hat{\theta} = \frac{\mu_0 n I r}{2} \hat{\theta} \text{ for } \le R \\ &= \frac{\mu_0 n I \pi R^2}{2\pi r} \hat{\theta} = \frac{\mu_0 n I R^2}{2r} \hat{\theta} \text{ for } \ge R \end{aligned}$$

where $\hat{\theta}$is the unit vector along the tangential direction.

Exercise 2

Obtain an expression for the vector potential inside a cylindrical wire of radius Rcarrying a current I.

(Ans. $-\mu_0 I r^2 / 4\pi R^2$)

The existence of a vector potential whose curl gives the magnetic field directly gives

$$\text{div } B = 0$$

as the divergence of a curl is zero. The vector identity

$$\nabla \times \vec{B} = \nabla \times \nabla \times \vec{A} = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$$

can be used to express Ampere's law in terms of vector potential. Using a Coulomb gauge in which $\nabla \cdot \vec{A} = 0$, the Ampere's law $\nabla \times \vec{B} = -\mu_0 \vec{J}$ is equivalent to

$$\nabla^2 \vec{A} = -\mu_0 \vec{J}$$

which is actually a set of three equations for the components of $vecA,$ viz.,

$$\begin{aligned} \nabla^2 A_x &= -\mu_0 J_x \\ \nabla^2 A_y &= -\mu_0 J_y \\ \nabla^2 A_z &= -\mu_0 J_z \end{aligned}$$

which are Poisson's equations.